From Hitchin systems to quantum groups

Joint works with Jakub Löwit and Shon Ngo

assisted by SandboxHQ=ChatGPT/Claude + Julia/OSCAR + Lean/Mathlib

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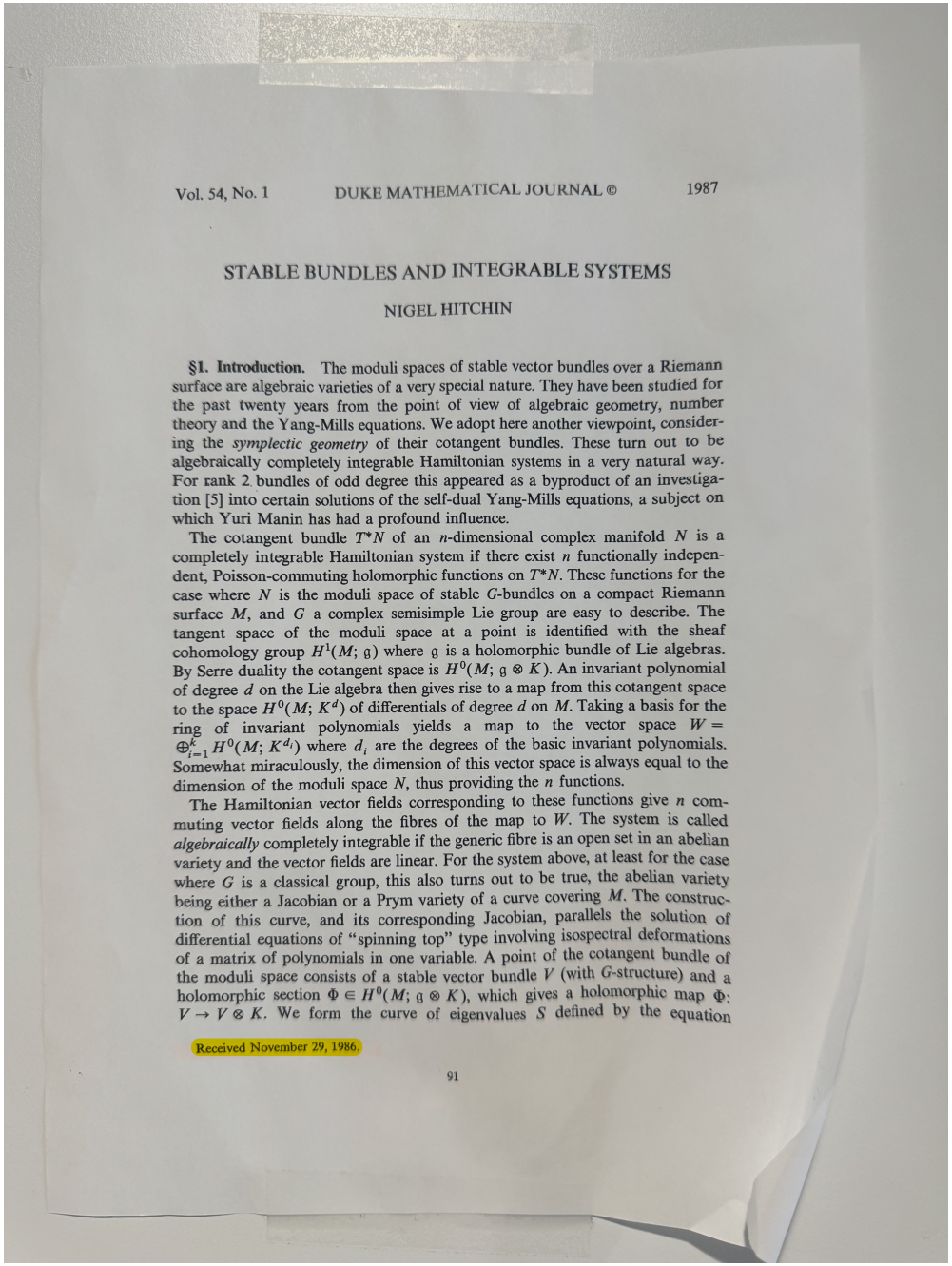
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STABLE BUNDLES AND INTEGRABLE SYSTEMS

NIGEL HITCHIN

§1. Introduction. The moduli spaces of stable vector bundles over a Riemann surface are algebraic varieties of a very special nature. They have been studied for the past twenty years from the point of view of algebraic geometry, number theory and the Yang-Mills equations. We adopt here another viewpoint, considering the *symplectic geometry* of their cotangent bundles. These turn out to be algebraically completely integrable Hamiltonian systems in a very natural way. For rank 2 bundles of odd degree this appeared as a byproduct of an investigation [5] into certain solutions of the self-dual Yang-Mills equations, a subject on which Yuri Manin has had a profound influence.

The cotangent bundle T^*N of an n-dimensional complex manifold N is a completely integrable Hamiltonian system if there exist n functionally independent, Poisson-commuting holomorphic functions on T^*N. These functions for the case where N is the moduli space of stable G-bundles on a compact Riemann surface M, and G a complex semisimple Lie group are easy to describe. The tangent space of the moduli space at a point is identified with the sheaf cohomology group $H^1(M; \mathfrak{g})$ where $\mathfrak{g}$ is a holomorphic bundle of Lie algebras. By Serre duality the cotangent space is $H^0(M; \mathfrak{g} \otimes K)$. An invariant polynomial of degree d on the Lie algebra then gives rise to a map from this cotangent space to the space $H^0(M; K^d)$ of differentials of degree d on M. Taking a basis for the ring of invariant polynomials yields a map to the vector space $W = \oplus_{i=1}^{k} H^0(M; K^{d_i})$ where d_i are the degrees of the basic invariant polynomials. Somewhat miraculously, the dimension of this vector space is always equal to the dimension of the moduli space N, thus providing the n functions.

The Hamiltonian vector fields corresponding to these functions give n commuting vector fields along the fibres of the map to W. The system is called *algebraically* completely integrable if the generic fibre is an open set in an abelian variety and the vector fields are linear. For the system above, at least for the case where G is a classical group, this also turns out to be true, the abelian variety being either a Jacobian or a Prym variety of a curve covering M. The construction of this curve, and its corresponding Jacobian, parallels the solution of differential equations of "spinning top" type involving isospectral deformations of a matrix of polynomials in one variable. A point of the cotangent bundle of the moduli space consists of a stable vector bundle V (with G-structure) and a holomorphic section $\Phi \in H^0(M; \mathfrak{g} \otimes K)$, which gives a holomorphic map $\Phi: V \to V \otimes K$. We form the curve of eigenvalues S defined by the equation

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De Concini–Procesi positive Borel for $\mathrm{U}_q(\mathrm{GL}_n)$

- (Drinfeld/Jimbo, 1985) $\mathrm{U}_{\mathbb{F}}^{\geq 0}$, $\mathbb{F} := \mathbb{Q}(q)$: generated by K_λ^+, $\lambda \in \mathbb{Z}^n$, and by E_i, $1 \leq i < n$, subject to

$$K_\lambda^+ K_\mu^+ = K_{\lambda+\mu}^+, \quad K_\lambda^+ E_i K_{-\lambda}^+ = q^{(\lambda,\alpha_i)} E_i, \quad E_i E_j = E_j E_i \ \ (|i-j|>1),$$

$$E_i^2 E_j - (q+q^{-1}) E_i E_j E_i + E_j E_i^2 = 0 \quad (|i-j|=1).$$

- with Hopf algebra structure, $\Delta : \mathrm{U}_{\mathbb{F}}^{\geq 0} \to \mathrm{U}_{\mathbb{F}}^{\geq 0} \otimes \mathrm{U}_{\mathbb{F}}^{\geq 0}$ coproduct, $\epsilon : \mathrm{U}_{\mathbb{F}}^{\geq 0} \to \mathbb{F}$ counit and $S : \mathrm{U}_{\mathbb{F}}^{\geq 0} \to \mathrm{U}_{\mathbb{F}}^{\geq 0}$ antipode:

$$\Delta(K_\lambda^+) = K_\lambda^+ \otimes K_\lambda^+, \qquad \Delta(E_i) = 1 \otimes E_i + E_i \otimes K_{\alpha_i}^+,$$

$$\epsilon(K_\lambda^+) = 1, \ \ \epsilon(E_i) = 0, \qquad S(K_\lambda^+) = K_{-\lambda}^+, \ \ S(E_i) = -E_i K_{-\alpha_i}^+.$$

- De Concini–Procesi (1993) integral form: $\mathrm{U}_{\mathbb{A}}^{\geq 0} \subset \mathrm{U}_{\mathbb{F}}^{\geq 0}$ the $\mathbb{A} := \mathbb{Z}[q^{\pm 1}]$-subalgebra generated by the K_λ^+ and by $e_{ab} := (q-q^{-1})E_{ab}$, $1 \leq a < b \leq n$, where $E_{i,i+1} := E_i$ and, for $b > a+1$, $E_{ab} := E_{a,b-1}E_{b-1,b} - q^{-1}E_{b-1,b}E_{a,b-1}$.

The extended Lusztig negative Borel

- V finite free $\mathbb{A}$-module with $\mathrm{U}_{\mathbb{A}}^{\geq 0}$-action, *type 1*:

$$V_{\mathbb{F}} := V \otimes_{\mathbb{A}} \mathbb{F} = \bigoplus_{\lambda \in \mathbb{Z}^n} (V_{\mathbb{F}})_{K_a^+ = q^{\lambda_a}}.$$

- For $m \in V$, $\ell \in V^* = \mathrm{Hom}_{\mathbb{A}}(V, \mathbb{A})$, put $c_{\ell,m}(x) = \ell(xm)$,

$$\mathrm{U}_{\mathbb{A}}^{\leq 0} := \mathrm{span}_{\mathbb{A}}\{c_{\ell,m}\} \subseteq \mathrm{Hom}_{\mathbb{A}}\Big(\mathrm{U}_{\mathbb{A}}^{\geq 0}, \mathbb{A}\Big).$$

  Its coefficient Hopf structure and pairing $\tau(x, \phi) = \phi(x)$ are

$$(\phi\psi)(x) = (\phi \otimes \psi)(\Delta x), \qquad (\Delta\phi)(x \otimes y) = \phi(yx).$$

- $\mathrm{U}_{\mathbb{A}}^{\leq 0}$ generated by modified divided powers (Lusztig, 1993):

$$F_i^{(r)} := \frac{F_i^r}{\prod_{s=1}^{r} \frac{1-q^{-2s}}{1-q^{-2}}}, \qquad (K_a^-)^{\pm 1}, \qquad \prod_{j=0}^{r-1} \frac{K_a^- - q^j}{q^r - q^j},$$

  where $F_i := \sum_{\lambda \in \mathbb{Z}^n} (K_\lambda^+ e_i)^*$ and $1 \leq i < n$, $1 \leq a \leq n$, $r \geq 1$.

Hybrid family quantum group is the Drinfeld double

- *Hopf pairing* $\tau : \mathrm{U}_{\mathbb{A}}^{\geq 0} \times \mathrm{U}_{\mathbb{A}}^{\leq 0} \to \mathbb{A}$, $\tau(x, \phi) = \phi(x)$:

$$\tau(K_\lambda^+, K_\mu^-) = q^{-\langle \lambda, \mu \rangle}, \quad \tau(e_i, F_j) = \delta_{ij}, \quad \tau(e_i, K_\mu^-) = 0 = \tau(K_\lambda^+, F_j).$$

- *Drinfeld double* $\mathbb{U} := \mathrm{U}_{\mathbb{A}}^{\geq 0} \otimes_{\mathbb{A}} \mathrm{U}_{\mathbb{A}}^{\leq 0}$, $\mathbb{A}$-Hopf algebra:

$$[e_i, F_j] = \delta_{ij}\left(K_{\alpha_i}^+ - K_{\alpha_i}^-\right), \qquad [K_\lambda^+, K_\mu^-] = 0,$$

  hybrid family quantum group (Hausel–Löwit, 2026).

- $t_a := K_a^+ K_a^-$ central group-like,
  $\mathbb{A}_t := \mathbb{A}[t_a, t_a^{-1}]_{1 \leq a \leq n} \subset \mathbb{U}$ central family Hopf torus.
- The Drinfeld double $\mathbb{U}_{\mathbb{F}} = \mathrm{U}_{\mathbb{F}}^{\geq 0} \otimes_{\mathbb{F}} \mathrm{U}_{\mathbb{F}}^{\leq 0}$ was considered by (Berenstein–Greenstein, 2017) and (Lacabanne, 2018); at $t_a = 1$ it gives (Drinfeld/Jimbo, 1985).
- $\mathbb{U}|_{t_a=1}$: DCP positive roots, Lusztig negative divided powers, extended Newton torus;
  close to the hybrid quantum groups of (Habiro–Lê, 2016), (Situ, 2022).

Verma characterisation and Harish–Chandra theorem

- *Family Verma module* $\mathbb{M}_{\mathbb{F}}(\lambda) := \mathbb{U}_{\mathbb{F}} \otimes_{\mathrm{U}_{\mathbb{F}}^{\geq 0}[(K_a^-)^{\pm 1}]} \mathbb{Q}(q)[t_a^{\pm 1}]\, m_\lambda$, $e_i m_\lambda = 0$, $K_a^- m_\lambda = q^{-\lambda_a} m_\lambda$, $K_a^+ m_\lambda = t_a q^{\lambda_a} m_\lambda$; lattice $\mathbb{M}(\lambda) := \mathbb{A}_t \mathrm{U}_{\mathbb{A}}^{\leq 0} m_\lambda$.
- $\mathrm{HC} : \mathbb{U}_{\mathbb{F}} \to \mathbb{U}_{\mathbb{F}}^0 = \mathbb{F}[(K_a^{\pm})^{\pm 1}]$ root-free PBW projection; $Y_a := q^{2(n-a)} K_a^+ (K_a^-)^{-1}$, $D := K_1^- \cdots K_n^-$; on m_λ: $\mathrm{ev}_\lambda(Y_a) = t_a q^{2(\lambda_a + n - a)}$, $\mathrm{ev}_\lambda(D) = q^{-\sum_a \lambda_a}$.

Theorem (Hausel–Löwit, 2026)

1. $\mathbb{U} = \left\{ x \in \mathbb{U}_{\mathbb{F}} : \ x\mathbb{M}(\lambda) \subseteq \mathbb{M}(\lambda) \ \ \forall \lambda \in \mathbb{Z}^n \right\}$.
2. $\mathrm{HC} : \ Z(\mathbb{U}) \xrightarrow{\sim} \mathbb{F}[t_a^{\pm 1}, D^{\pm 1}, Y_a^{\pm 1}]^W \cap \mathbb{U} \subset \mathbb{U}_{\mathbb{F}}$.

- Proof via rational HC, Verma separation + (Drinfeld, 1990): $Z(\mathbb{U}_{\mathbb{F}}) \cong \mathbb{Q}(q)[t_a^{\pm 1}, D^{\pm 1}, e_1(Y), \ldots, e_{n-1}(Y)]$; $R_\tau = \sum_i x_i \otimes y_i$ the canonical element of τ, $V = \Lambda_q^r \mathbb{F}^n$, $M_V := (1 \otimes \rho_V)(R_{\tau,21} R_\tau)$ commutes with $(1 \otimes \rho_V)\Delta(\mathbb{U}_{\mathbb{F}})$; $C_r := q^{r(n-1)} (1 \otimes \mathrm{Tr}_V)\left[(1 \otimes \rho_V(K_{2\rho}^+)) M_V \right]$ central, $\mathrm{HC}(C_r) = e_r(Y)$.

Equivariant K-theory of Gr^μ

- G complex reductive, X complex projective variety, $\mathrm{G} \curvearrowright X$
- for $\mathrm{G} = \mathrm{GL}_n$: $\mathrm{Gr} := \mathrm{GL}_n(\mathbb{C}((u)))/\mathrm{GL}_n(\mathbb{C}[[u]])$ *affine Grassmannian*, $\mathrm{Gr}^\mu := \overline{\mathrm{GL}_n(\mathbb{C}[[u]])\, u^\mu}$ *affine Schubert variety*, for μ dominant; $\mathrm{G}^\vee = \mathrm{GL}_n(\mathbb{C})$ acts on the left, $\mathbb{C}^\times_{\mathrm{rot}}$ by $u \mapsto s^2 u$; $T^\vee \subset \mathrm{G}^\vee$ torus, $(\mathrm{Gr}^\mu)^{T^\vee} = wt(\mu) := \{\text{weights of } V^\mu\}$
- (Löwit 2025) defines *fixed point scheme* $\mathrm{Fix}_{\mathrm{G}}(X) : R \mapsto \{(g, x) \in \mathrm{G}(R) \times X(R) \mid gx = x\}$. develops $K_{\mathrm{G}}(X; \mathbb{C}) \cong \mathbb{C}[\mathrm{Fix}_{\mathrm{G}}]^{\mathrm{G}}$ for a nice class of varieties
- (Löwit 2026) proves (Bezrukavnikov 2023) conjecture: $K_{T^\vee \times \mathbb{C}^\times}(\mathrm{Gr}^\mu; \mathbb{C}) = \mathbb{C}[z_a^{\pm 1}, q^{\pm 1}]\langle b_{i,\ell} \rangle_{\ell > 0}^{1 \le i \le n}$ where $b_{i,\ell}$ are the *Bezrukavnikov classes*
- $H := T^\vee \times \mathbb{C}^\times_{\mathrm{rot}}$, $q(s) = s$; GKM description

$$K_H(\mathrm{Gr}^\mu; \mathbb{C}) \xrightarrow[\cong]{\mathrm{GKM}^\mu} \Big\{ (f_\lambda)_\lambda \in \bigoplus\nolimits_{\lambda \in wt(\mu)} \mathbb{C}[z_a^{\pm 1}, q^{\pm 1}] : \Big(1 - \tfrac{z_a}{z_b} q^{(\lambda+\nu)_a - (\lambda+\nu)_b}\Big) \mid (f_\lambda - f_\nu) \text{ for } \lambda - \nu \in \mathbb{Z}(\varepsilon_a - \varepsilon_b),\ a < b \Big\}$$

- $\mathrm{GKM}^\mu(b_{1,\ell})_\lambda = \dfrac{\sum_a z_a^\ell - \sum_a z_a^\ell q^{2\ell\lambda_a}}{1 - q^{2\ell}} \in \mathbb{C}[z_a^{\pm 1}, q^{\pm 1}]$

Completed centre is equivariant K-theory

- $\chi_\lambda := \mathrm{ev}_\lambda \circ \mathrm{HC} : Z(\mathbb{U}) \to \mathbb{A}_t$, the scalar by which $Z(\mathbb{U})$ acts on $\mathbb{M}(\lambda)$. *Verma windows* $wt(\mu) \subset \mathbb{Z}^n$:

$$J_\mu := \bigcap\nolimits_{\lambda \in wt(\mu)} \mathrm{Ann}_{\mathbb{U}} \mathbb{M}(\lambda), \quad I_\mu := Z(\mathbb{U}) \cap J_\mu = \bigcap\nolimits_\lambda \ker \chi_\lambda;$$

$$\widehat{\mathbb{U}} := \varprojlim_\mu \mathbb{U}/J_\mu, \qquad \widehat{Z}(\mathbb{U}) := \varprojlim_\mu Z(\mathbb{U})/I_\mu.$$

Theorem (Hausel–Löwit, 2026)

Fixed-point restriction gives

$$\widehat{Z}(\mathbb{U}) \xrightarrow{\sim} Z(\widehat{\mathbb{U}}) \xrightarrow[\sim]{\Theta} K_H(\mathrm{Gr}), \qquad i_\lambda^* \Theta(C) = \chi_\lambda(C).$$

- $B_\ell := \Theta^{-1}(b_{1,\ell}) \in Z(\mathbb{U})$ the *Bezrukavnikov classes*; by (Löwit, 2026) $Z(\mathbb{U}_\mathbb{F}) = \mathbb{F}[t_a^{\pm 1}][D^{\pm 1}, B_\ell]_{\ell \in \mathbb{Z}_{>0}}$.
- Their *elementary counterparts* by Newton identities: $E_0 = 1$, $E_m = \frac{1}{m} \sum_{\ell=1}^{m} (-1)^{\ell-1} B_\ell E_{m-\ell}$ are actually in $Z(\mathbb{U})$. *elementary Bezrukavnikov classes*

Quantum medium algebras and the affine Hecke centre

- V^μ Weyl lattice of $\mathbb{U}|_{t_a=1}$ for $\mu_1 \geq \cdots \geq \mu_n \geq 0$;
  $z \in Z(\mathbb{U})$ acts on $\mathbb{M}(0) \otimes_{\mathbb{A}} V^\mu$ by $\chi_0(z)$ and by $\Delta_\mu(z)$.
  $$\mathcal{Z}_q^\mu := \Big(\mathbb{A}_t\Big[\Delta_\mu(z) : z \in Z(\mathbb{U})\Big]\Big)^W \subseteq \mathrm{End}_{\mathbb{U}}\Big(\mathbb{M}(0) \otimes_{\mathbb{A}} V^\mu\Big),$$
  Quantum medium algebra.
- (Orellana–Ram, 2007) On $\mathbb{M}(0) \otimes V_{\mathrm{st}}^{\otimes d}$: $T_i := P_{i,i+1} R_{i,i+1}^{V_{\mathrm{st}},V_{\mathrm{st}}}$,
  $X_1 := q^{2(n-1)} R_{21} R_{12}$, $X_{i+1} := T_i X_i T_i$; braid relations,
  $(T_i - q)(T_i + q^{-1}) = 0$, $[X_i, X_j] = 0$; they commute with $\mathbb{U}$
  $\rightsquigarrow \rho_d : \mathcal{H}_d^{\mathrm{aff}} \to \mathrm{End}_{\mathbb{U}}\Big(\mathbb{M}(0) \otimes V_{\mathrm{st}}^{\otimes d}\Big)$ an action of the *affine Hecke algebra* $\mathcal{H}_d^{\mathrm{aff}}$
- *Bernstein centre*:
  $Z(\mathcal{H}_d^{\mathrm{aff}}) = \mathbb{A}[X_1^{\pm 1}, \ldots, X_d^{\pm 1}]^{S_d} = \mathbb{A}[e_1(X), \ldots, e_{d-1}(X), e_d(X)^{\pm 1}]$.

- $Z(\mathcal{H}_{|\mu|}^{\mathrm{aff}}) \xrightarrow{\rho_{|\mu|}} \mathrm{End}_{\mathbb{U},\mathcal{H}_{|\mu|}}\Big(\mathbb{M}(0) \otimes V_{\mathrm{st}}^{\otimes|\mu|}\Big) \xrightarrow{\mu\text{-cell}} \mathrm{End}_{\mathbb{U}}\Big(\mathbb{M}(0) \otimes V^\mu\Big)$

Integral generators for quantum medium algebras

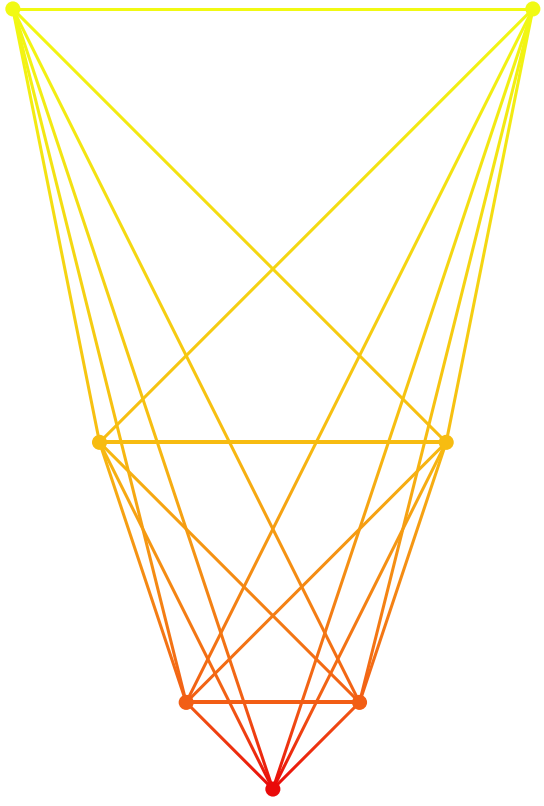
Theorem (Hausel–Löwit–Ngo, 2026)

1. $z \mapsto (\chi_\lambda(z))_{\lambda \in \mathrm{wt}(\mu)}$ *gives*
$$\mathcal{Z}_q^\mu \xrightarrow{\sim} K_{\mathrm{G}^\vee \times \mathbb{C}^\times_{\mathrm{rot}}}(\mathrm{Gr}^\mu).$$

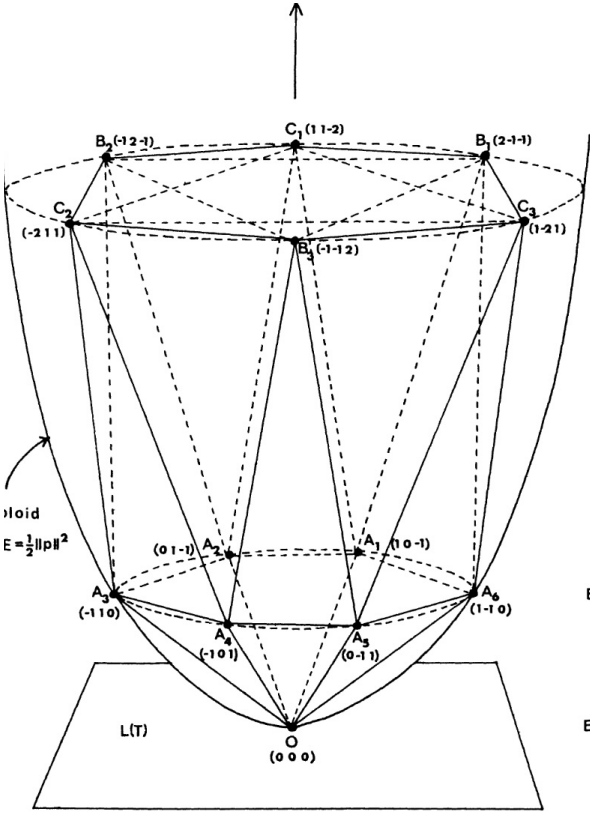
2. $\mathrm{Gr}^\mu \subseteq \{L \subseteq \mathbb{C}[[u]]^n : \dim \mathbb{C}[[u]]^n/L = |\mu|\}$, *Q the tautological rank-*$|\mu|$ *quotient:*
$$\begin{array}{ccccc} \mathbb{A}_t^W \otimes_{\mathbb{A}} Z(\mathcal{H}^{\mathrm{aff}}_{|\mu|}) & \twoheadrightarrow & \mathcal{Z}_q^\mu & \xrightarrow{\sim} & K_{\mathrm{G}^\vee \times \mathbb{C}^\times_{\mathrm{rot}}}(\mathrm{Gr}^\mu) \\ e_m(X) & \mapsto & \Delta_\mu(E_m) & \leftrightarrow & [\Lambda^m Q] \end{array}$$

3. $\mathcal{Z}_q^\mu = \mathbb{A}_t^W[E_1, \ldots, E_{|\mu|}]$.

Moment polytope and GKM graph

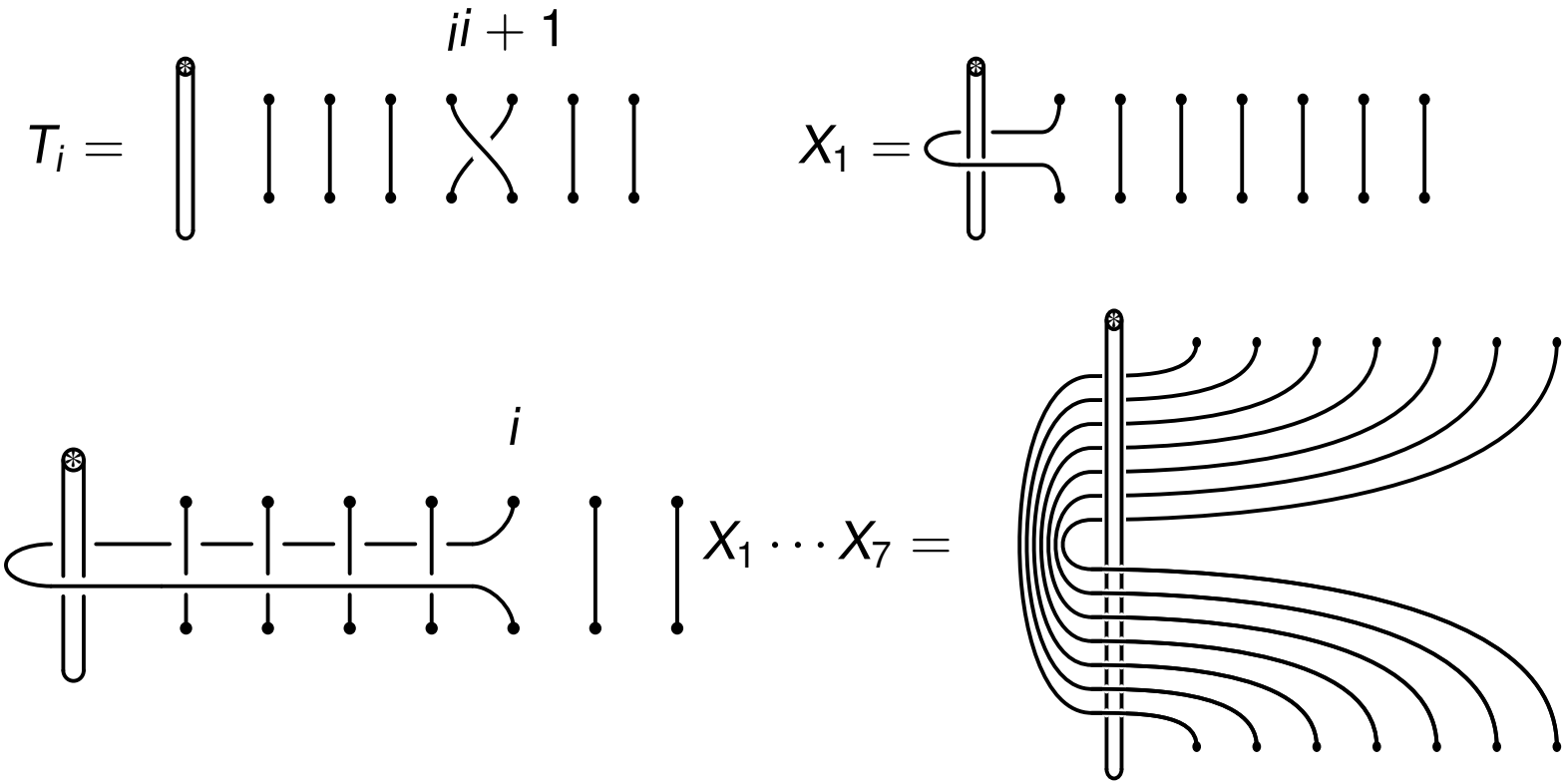


$\mathrm{Gr}_{\mathrm{SL}_2}$ (Harada–Henriques–Holm 2006)



$\Omega\mathrm{SU}(3)$ (Atiyah–Pressley 1983)

Affine braid pictures



Connections to the Hitchin system

- *Dol–NAH*: classical medium algebra $\mathcal{M}^\mu \leftrightarrow$ upward flows (Hausel–Hitchin, 2022; Hausel, 2023–24; González, 2026):

  $H^{2*}_{\mathrm{G}^\vee}(\mathrm{Gr}^\mu;\mathbb{C}) \cong \mathcal{M}^\mu \rightsquigarrow$ Hecke transformations for $(W^+_\mu \xrightarrow{h} \mathcal{A})$.

- *Hodge–NAH*: Rees-filtered medium algebras (Hausel, 2025):

  $$H^{2*}_{\mathrm{G}^\vee\times\mathbb{C}^\times_{\mathrm{rot}}}(\mathrm{Gr}^\mu;\mathbb{C}) \cong \mathcal{Z}^\mu_\hbar \overset{?}{\rightsquigarrow} \text{ Hecke transformations.}$$

- *Quantum Betti–NAH?*: q-difference NAH (Mochizuki, 2019):

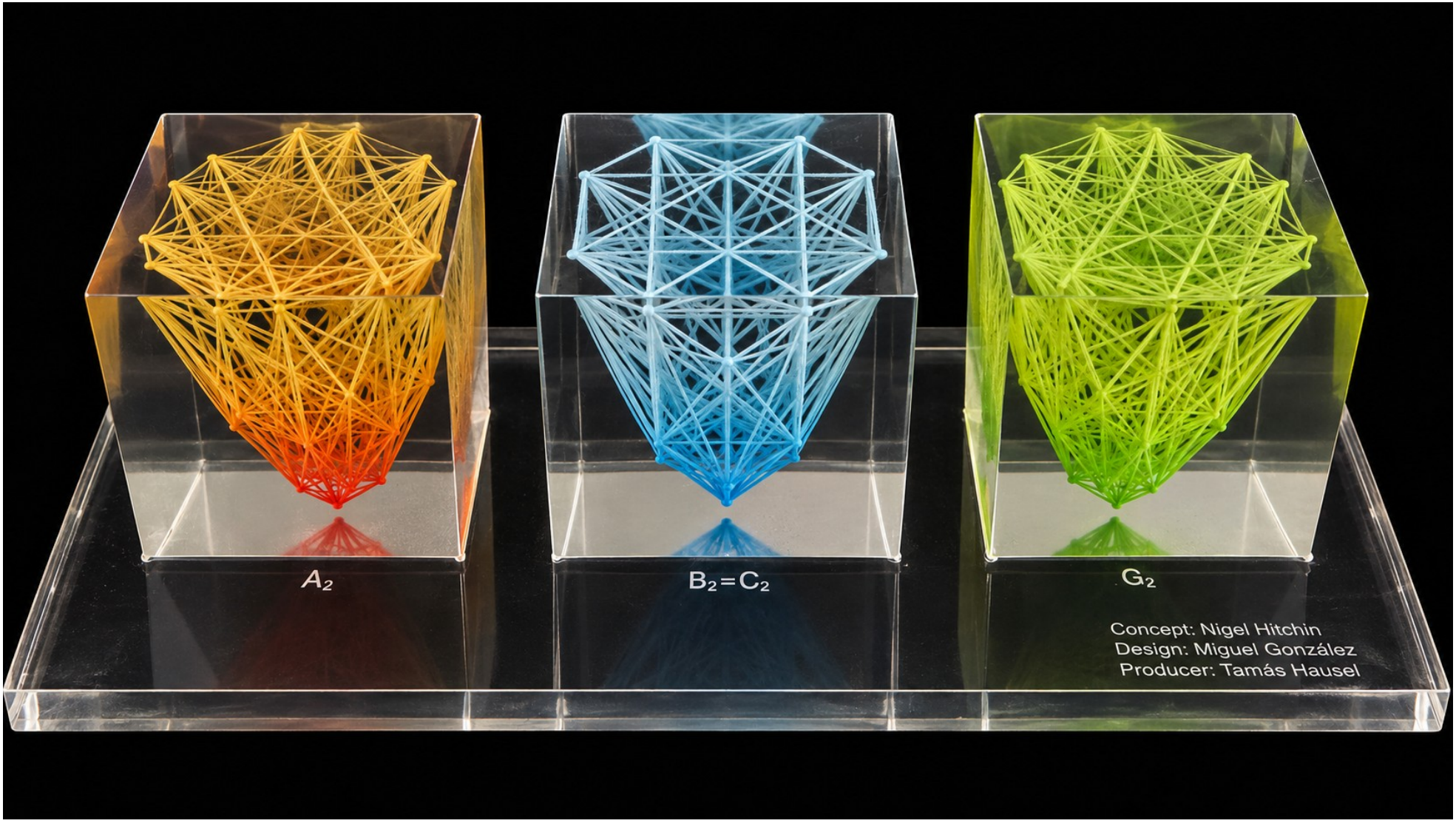
  $$K_{\mathrm{G}^\vee\times\mathbb{C}^\times_{\mathrm{rot}}}(\mathrm{Gr}^\mu) \cong \mathcal{Z}^\mu_q \overset{?}{\rightsquigarrow} \text{ Hecke transformations.}$$

  At $q{=}1$: Hecke transformations for multiplicative Hitchin systems?

- *Cohomological limit*: $\mathfrak{J} := (q-1,\ t_a-1,\ e_\beta : \beta > 0) \subset \mathbb{U}$ two-sided, $\deg \mathfrak{J}/\mathfrak{J}^2 = 2$:

  $$\mathbb{U}_\hbar := \mathrm{gr}_{\mathfrak{J}}\,\mathbb{U}, \qquad Z(\mathbb{U}_{\hbar,c}) \overset{?}{\cong} H^{2*}_H(\mathrm{Gr}_c;\mathbb{Z}).$$

Tip of the iceberg of the Hitchin system



Boldog születésnapot Nigel!

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